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Path sensitivity and uncertainty propagation in SEA

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Abstract

This work addresses the differentiation of the inverse coupling matrix with respect to a coupling or damping loss factor. It applies to two central questions in conjunction with statistical energy analysis (SEA). The first concerns optimization problems for which a sensitivity factor is proposed, based on a simple Taylor series. The factor shows the sensitivity of the vibrational energy flow to a certain coupling loss factor and hence the associated path.

The second question where the differentiated inverse coupling matrix applies is the investigation of the propagation of uncertainties in input quantities in a statistical energy analysis. This enables a variance calculation for primary the energies of the subsystems. The approach can be used to establish a variation range for the results from an SEA analysis.

Generic configurations with different transmission paths are investigated employing numerical implementations of the approaches. Both procedures are applicable for systems of arbitrary size.

1. Introduction

Statistical energy analysis (SEA) is an efficient tool for the analysis of vibro-acoustic systems. SEA provides good results under the conditions of high-modal overlap, randomly spaced resonances and that the energy is distributed equally over the modes of vibration of a subsystem.

This paper splits in two parts employing the derivative of the SEA coupling matrix. The first is the optimization of vibro-acoustic systems with respect to the energy flow for which an influence or a sensitivity factor is proposed. The second is the estimation of the propagation of uncertainties of the system parameters in the analysis.

1.1. Optimization

The application of statistical energy analysis in noise and vibration control is frequently reported in the literature. In Ref. [1] Davis states that: ''Noise control is based on identifying and then reducing energy flow paths. This can be done by back tracing of energy from receiver to source or Craik's coupling to total loss

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Nomenclature

C	coupling matrix Eq. (3)	W	transmitted power Eq. (2)
c	elements of C	x_u	uncertain input variable
E	energy	σ	variance Eq. (18)
N	number of modes	η_{Di}	damping loss factor
P	path Eq. (1)	η_{ii}	total loss factor $\eta_{ii} = \eta_{Di} + \sum_j \eta_{ij}$
$S_{i,kl}$	sensitivity Eq. (8)	η_{ij}	coupling loss factor
		ω	angular frequency

factor product ratio''. Craik defines in Ref. [2] the path P along subsystems 1,2,3 to n as

$$P_{1-2-3-\cdots-n} = \prod_{i=1}^{n-1} \frac{\eta_{ii+1}}{\eta_{ii}}, \tag{1}$$

where η_{ij} is the coupling loss factor (CLF) and $\eta_{ii} = \eta_{Di} + \sum_j \eta_{ij}$ is called total loss factor. The CLF η_{ij} can be defined by the power W_{ij} transmitted from subsystem i to subsystem j

$$W_{ij} = E_i \omega \eta_{ij}. \tag{2}$$

Because both methods have practical and theoretical problems, Davis concludes that: ''More intelligent path identification and assessment algorithms are needed.''

This work takes up on this need and proposes an alternative approach. To establish a basis for an improved vibro-acoustic design of a certain system the question can be posed: How much do changes of certain elements influence the vibrational energy in a particular subsystem? To answer this question the energy $E_j(\eta_{kl})$ in a subsystem j is considered a function of the CLFs η_{kl}. By means of the derivatives $\partial E_j/\partial \eta_{kl}$, it is possible to get an overview of which CLFs are most likely to change the vibrational energy in a given subsystem. This is termed the sensitivity of a subsystem energy to the CLF. Analogous analyses can be undertaken for every variable affecting the vibrational energy flow.

1.2. Propagation of uncertainties

Lyon [3,4] describes a method to assess the variance or confidence of results from a statistical energy analysis. A population is considered of similar systems with slightly different dimensions and a standard deviation is derived. The results support the conventional assumption that the confidence of the SEA prediction increases with frequency. He concludes, however, that ''There is a considerable area of interesting research work that needs to be done in analysing variance of interacting systems.''

A continuation of this work is done in a more recent study by Langley and Cotoni [5]. The study is based on the Gaussian orthogonal ensemble, detailed in Weaver [6].

Culla et al. [7,8] broaden the range of uncertain variables to the system parameters (e.g. mass, stiffness and dimensions). A numerical study is undertaken of the confidence factor by using a Monte Carlo simulation technique for several general systems. It is found that the results do not exhibit the same tendency as those predicted by Lyon. Accordingly, and in line with Lyon's statement, the topic deserves a deeper and more detailed investigation.

de Langhe [9], suggested a similar procedure which is employed in the context of the power injection method and finds comparable results.

In this study also an analytical approach is proposed for the investigation of the problem of uncertain system parameters and the propagation of the uncertainties in calculations of the subsystems energies. A similar approach is employed in a standard [10] for the general assessment of the compound error of quantities based on multiple, measured values with uncertainties. The analyses with partial derivatives show the impact of a certain parameter uncertainty and can also be used to quantify the impact of the parameter itself.

2. Theory

Consider a system composed of N subsystems with their energies E_i, the CLF η_{ij} and the power inputs P_i. The energy-flux in an SEA model of the system is given by [11]

$$\omega \begin{pmatrix} \eta_{D1} + \sum_{i=2}^{N} \eta_{1i} & \cdots & -\eta_{N1} \\ \vdots & \ddots & \vdots \\ -\eta_{1N} & \cdots & \eta_{DN} + \sum_{i=1}^{N-1} \eta_{Ni} \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_N \end{pmatrix} = \begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix}, \tag{3}$$

where η_{Di} are the ordinary loss factors due to damping. In matrix notation, Eq. (3) reads

$$\omega \mathbf{C} \cdot \mathbf{E} = \mathbf{P}, \tag{4}$$

where $\mathbf{C}$ is the SEA or coupling matrix with elements c_{ij}. In the following $\mathbf{P}$ will be replaced by the energy quantity $\mathbf{G_P} = \mathbf{P}/\omega$ and Eq. (4) becomes

$$\mathbf{C} \cdot \mathbf{E} = \mathbf{G_P}. \tag{5}$$

This means that the column vector of subsystem energies can be expressed as

$$\mathbf{E} = \mathbf{C^{-1}} \cdot \mathbf{G_P}. \tag{6}$$

2.1. Sensitivity to CLFs

The first element of the Taylor series for the element of the energy E_j is given by

$$E_j(\eta_{kl}) = E_j(\hat{\eta}_{kl}) + \sum_k \sum_{l>k} \left. \frac{\partial E_j}{\partial \eta_{kl}} \right|_{\hat{\eta}} (\eta_{kl} - \hat{\eta}_{kl}), \tag{7}$$

which is a function of all η_{kl} at the expansion point $\hat{\eta}$. The associated CLFs η_{lk} are related to η_{kl} via the reciprocity relationship $\eta_{lk} n_k = \eta_{kl} n_l$. The analysis therefore will be carried out only for $l > k$.

In principle, it is possible to analyse the oppositely directed CLFs separately. However, Craik [12] finds in a analysis of random errors in CLFs that ignoring the reprocity relationship and assuming an independent η_{lk} with the same error result in a greater deviation. The assumed physical effect is that the change of the CLF is partly compensated by the dependent oppositely directed counterpart. Since the resulting error and the sensitivity are highly interrelated as will be seen later, it is assumed that a separate analysis would overestimate the effect of a change in a CLF. Instead, a sensitivity factor is sought that indicates where a change in a CLF would have the largest repercussions. The sensitivity factor $S_{j,kl}$ is the sensitivity of an energy E_j due to changes of the CLF η_{kl}.

Especially to compare different sensitivity factors, it is assumed that a change of the CLF η_{kl} will be of the order of the value $\hat{\eta}_{kl}$ at maximum. The value $\hat{\eta}_{kl}$ at maximum is taken because the Taylor series is expanded at this point and the changes will be small relative to the expansion point. It follows that the sensitivity (factor) is defined as

$$S_{j,kl} = \frac{1}{E_j} \left. \frac{\partial E_j}{\partial \eta_{kl}} \right|_{\hat{\eta}} \hat{\eta}_{kl}. \tag{8}$$

This is a reasonable estimate for the assessment of the influence of small variations in η_{kl}. The derivative $\partial E_j / \partial \eta_{kl}$ for an arbitrary SEA system will be calculated in Section 2.1.1.

Upon considering Eq. (7), the estimated effect of a change can be calculated, without solving the whole system again i.e.,

$$\Delta E_j = E_j S_{j,kl} \frac{\hat{\eta}_{kl} - \eta_{kl}}{\eta_{kl}}. \tag{9}$$

Greater changes of the CLFs and a more accurate effect estimation can be achieved by including more terms of the Taylor series (7), but herein a first-order indication of the influence is taken to be sufficient.

2.1.1. Partial derivative of an inverse matrix with respect to its components

The partial derivatives of the energy E_j with respect to the CLF or any uncertain input quantity x_u of subsystem u ($u \neq j$) is obtained from the derivatives of the energy in the jth subsystem with respect to its elements c_{kl} that are functions of x_u (usually c_{uk}, c_{uu}, c_{kk} and c_{ku}). By using the chain rule for functions of several variables

$$\frac{\partial E_j}{\partial x_u} = \sum_{l=1}^{N} \sum_{k=1}^{N} \frac{\partial E_j}{\partial c_{kl}} \frac{\partial c_{kl}}{\partial x_u}. \tag{10}$$

Since the energy E_j can be expressed as the product of the inverse coupling matrix and the energy G_P, $\mathbf{E} = \mathbf{C}^{-1} \cdot \mathbf{G_P}$, the partial derivatives of the inverse coupling matrix $\mathbf{C}^{-1}$ with respect to its—non inverted—elements c_{kl} will be examined. It should be noted that the partial derivative with respect to c_{kl} is not equal to that with respect to η_{lk}. Here an intermediate step is employed which makes it much easier to form the partial derivative $\partial(\mathbf{C}^{-1} \cdot \mathbf{G_P})_j / \partial \eta_{kl}$ or that with respect to any other variable x_u, with $\eta_{kl}(x_u)$.

At first a slightly modified variant of the common sub-matrix, that is used to build the inverse, is defined. This simplifies the calculation of the derivative. To build the sub-matrix $M_{\mu\nu}$ the (μ, ν) element of the matrix M is set to unity and the rest of the elements in the μ-row or the ν-column to zero. For details of the procedure and a proof that the sub-matrix is a result of the differentiation see Appendix A.

By employing the results given in the appendix, the partial derivatives $\partial E_j / \partial c_{kl}$ can be formed as

$$\mathbf{E} = \mathbf{C^{-1}} \mathbf{G_P},$$

$$E_j = \sum_{i}^{N} \frac{1}{\det \mathbf{C}} \det \mathbf{C_{ij}} G_{P_i},$$

$$\frac{\partial E_j}{\partial c_{kl}} = \sum_{i=1}^{N} \frac{\det \mathbf{C_{ij_{kl}}} G_{P_i}}{\det \mathbf{C}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C^2}} \det \mathbf{C_{ij}} G_{P_i}. \tag{11}$$

Moreover, the partial derivative of an energy with respect to c_{kl}, normalized with respect to the energy, is given by

$$\frac{1}{E_j} \frac{\partial E_j}{\partial c_{kl}} = \sum_{i=1}^{N} \frac{\det \mathbf{C}}{\det \mathbf{C_{ij}} G_{P_i}} \sum_{i}^{N} \left(\frac{\det \mathbf{C_{ij_{kl}}} G_{P_i}}{\det \mathbf{C}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C}^2} \det \mathbf{C_{ij}} G_{P_i} \right). \tag{12}$$

Provided that $G_{P_i} \neq 0$ this reduces to

$$\frac{1}{E_j} \frac{\partial E_j}{\partial c_{kl}} = \frac{\sum_i^N \det \mathbf{C_{ij_{kl}}} G_{P_i}}{\sum_i^N \det \mathbf{C_{ij}} G_{P_i}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C}}. \tag{13}$$

If $G_{P_i} = 0$, the ith element of the sum $\sum_{i=1}^{N}$ is zero.

To obtain the sensitivity for η_{kl}, the partial derivative $\partial c_{ij} / \partial \eta_{kl}$ is required,

$$\frac{\partial c_{ij}}{\partial \eta_{kl}} = \begin{cases} -\Delta N_l / \Delta N_k & \text{for } i = k \text{ and } j = l, \\ 1 & \text{for } i = j \text{ and } i = k, \\ -1 & \text{for } i = l \text{ and } j = k, \\ \Delta N_l / \Delta N_k & \text{for } i = j \text{ and } i = l, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The resulting expression for the sensitivity to a CLF η_{kl} thus can be developed from Eq. (8) to be given by

$$S_{j,kl} = \left(-\frac{\Delta N_l}{\Delta N_k} \frac{\sum_i^N \det \mathbf{C}_{\mathbf{ij_{kl}}} G_{P_i}}{\sum_i^N \det \mathbf{C}_{\mathbf{ij}} G_{P_i}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C}} + \frac{\sum_i^N \det \mathbf{C}_{\mathbf{ij_{kk}}} G_{P_i}}{\sum_i^N \det \mathbf{C}_{\mathbf{ij}} G_{P_i}} - \frac{\det \mathbf{C_{kk}}}{\det \mathbf{C}} - \frac{\sum_i^N \det \mathbf{C}_{\mathbf{ij_{lk}}} G_{P_i}}{\sum_i^N \det \mathbf{C}_{\mathbf{ij}} G_{P_i}} - \frac{\det \mathbf{C_{lk}}}{\det \mathbf{C}} + \frac{\Delta N_l}{\Delta N_k} \frac{\sum_i^N \det \mathbf{C}_{\mathbf{ij_{ll}}} G_{P_i}}{\sum_i^N \det \mathbf{C}_{\mathbf{ij}} G_{P_i}} - \frac{\det \mathbf{C_{ll}}}{\det \mathbf{C}} \right) \eta_{kl}. \tag{15}$$

2.1.2. Computational effort

In the case of many subsystems the computational effort is of great importance for the practicability of a method. In this section the order of magnitude of operations as a function of the number of subsystems N is compared with that of the path-by-path method. The maximum number of independent CLFs is $0.5(N-1)^2$. For each CLF a series of determinants need be computed, each of which with a maximum effort of the order of $O(N^3)$, assuming a Gaussian elimination. The resulting effort to compute the sensitivity factors, is therefore of the order of $O(N^5)$. It will take about N^2 the time of solving the SEA matrix itself. With this step performed however the effects of changes in the system can be estimated without solving the whole system again.

Usually the path-by-path analysis is done by selecting the dominating paths based on experience or intuition and comparing them. A reason is the computational effort of a complete analysis. In general, the maximum number p of paths of length n in an N subsystem model is $p = (N-2)^{n-2}$ [2]. In a real model, most of the CLFs will be zero, which reduces this number. By assuming that the longest path of interest is of the length N and the shortest has the length of one, it follows that

$$p_{tot} = \sum_{i=0}^{N-2} (N-2)^i = \frac{(N-2)^{N-1} - 1}{N-3}. \tag{16}$$

For each path, a product of a maximum of N factors need be calculated, according to Eq. (1), such that the overall effort is of the order $O(N^{N-1})$.

2.1.3. Path analysis and sensitivity factors

A simple example is presented to illustrate the relation between the path analysis and sensitivity factors. Consider a system consisting of two subsystems where only subsystem 1 is fed by the power P_1. It follows that the vibrational energy E_2 in subsystem 2 is $E_2 = (\eta_{12}/\eta_{22})E_1$. The path $P_{1-2} = \eta_{12}/\eta_{22}$. The sensitivity factor is

$$S_{2,12} = \frac{E_1}{E_2} \left(\frac{\eta_{12}}{\eta_{22}} + \frac{\eta_{12}\eta_{21}}{\eta_{22}^2} \right). \tag{17}$$

Upon assuming $\eta_{ij} \ll \eta_{ii}$, $\eta_{12}\eta_{21}/\eta_{22}^2$ can be neglected in comparison with η_{12}/η_{22} and it follows that $P_{1-2} \approx (E_2/E_1)S_{2,12}$. Provided the condition $\eta_{ij} \ll \eta_{ii}$ is valid for all subsystems this can be generalized for longer paths following the idea of Craik's path analysis as, for example, $P_{1-2-3} \approx E_3/E_1\, S_{3,12}S_{3,23}$.

2.2. Propagation of uncertainties

The CLFs, which have to be input by the analyst, are functions of the geometry and material properties. Henceforth, the input quantities will be denoted x_i. The uncertainties of the input quantities lead to uncertainties of the resulting energies characterized by a variance. A possibility to assess the resulting variance is to develop the functional relationship $E(x_i)$ and to determine the partial derivatives, as given by expression (18). The variance σ_f^2 of a function f that is depending on a large number of independent variables x_i is given approximately by Gauss' error propagation law. This means that for a limited number M of variables, the variance is approximated by

$$\sigma_f^2 = \sum_{i=1}^{M} \left(\frac{\partial f(x)}{\partial x_i} \right)^2 \sigma_{x_i}^2, \tag{18}$$

where $\sigma^2_{x_i}$ is the variance of the input quantity x_i. The latter variance can be based on an estimation or assessed from repeated (n_i) measurements of the variable x_i i.e., assessed from

$$\sigma^2_{x_i} = \frac{\sum_{l=1}^{n_i} (x_{il} - \overline{x_i})^2}{n_i(n_i - 1)}. \tag{19}$$

This would mean that a functional relationship would have to be developed for each SEA subsystem and each uncertain parameter. This effort can be circumvented by rewriting the derivatives, which gives the energy variance σ^2_E caused by the uncertainty of an input quantity x_u such that

$$\sigma^2_{E_j} = \left(\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial E_j}{\partial c_{kl}} \frac{\partial c_{kl}}{\partial x_u} \right)^2 \sigma^2_{x_u}. \tag{20}$$

Hence, the normalized standard deviation due to all uncertain input quantities x_u is given by

$$\frac{\sigma_{E_j}}{E_j} = \sqrt{\sum_{u=1}^{N} \left(\sum_{l=1}^{N} \sum_{k=1}^{N} \frac{1}{E_j} \frac{\partial E_j}{\partial c_{kl}} \frac{\partial c_{kl}}{\partial x_u} \right)^2 \sigma^2_{x_u}}. \tag{21}$$

Finally, the expression for the normalized standard deviation is found by inserting Eq. (13) in Eq. (21),

$$\frac{\sigma_{E_j}}{E_j} = \sqrt{\sum_{u=1}^{N} \left(\sum_{l=1}^{N} \sum_{k=1}^{N} \left(\frac{\sum_i^N \det \mathbf{C}_{\mathbf{ij_{kl}}} G_{P_i}}{\sum_i^N \det \mathbf{C}_{\mathbf{ij}} G_{P_i}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C}} \right) \frac{\partial c_{kl}}{\partial x_u} \right)^2 \sigma^2_{x_u}}. \tag{22}$$

Note that the uncertain parameter x_u represents a CLF or any other quantity. The term $\partial c_{kl}/\partial \eta_{kl}(x_u)$ will usually be non-zero only for the four occurrences $\{(k,l);(k,k);(l,k);(l,l)\}$ of η_{kl} in the coupling matrix $\mathbf{C}$.

3. Examples: sensitivity analysis

Consider a system composed of two subsystems where subsystem 1 is fed by the energy G_{P_1} and subsystem 2 is only indirectly excited i.e., $G_{P_2} = 0$. The CLFs are related by

$$\eta_{12} = \Delta N_2 / \Delta N_1 \eta_{21}. \tag{23}$$

For $N = 2$ and $G_{P_2} = 0$, Eq. (13) gives

$$\frac{1}{E_2} \frac{\partial E_2}{\partial c_{kl}} = \frac{\det \mathbf{C_{12_{kl}}}}{\det \mathbf{C_{12}}} - \frac{\det \mathbf{C_{kl}}}{\det \mathbf{C}} = \frac{1}{\det \mathbf{C}} \begin{pmatrix} -c_{22} & c_{21} \\ c_{12} + \det \mathbf{C}/c_{21} & -c_{11} \end{pmatrix}. \tag{24}$$

Upon expressing the SEA matrix elements in terms of the associated CLFs η, using Eq. (3) and applying Eq. (15) the sensitivity factor

$$\begin{aligned} S_{2,12} &= \left(\frac{\gamma\eta_{12} + \eta_{21} - (\eta_{D2} + \eta_{21}) - \gamma(\eta_{D1} + \eta_{12})}{(\eta_{D1} + \eta_{12})(\eta_{D2} + \eta_{21}) - \eta_{12}\eta_{21}} + \frac{1}{\eta_{12}} \right) \Bigg|_{\hat{\eta}} \hat{\eta_{12}} \\ &= 1 - \frac{\eta_{D2}\eta_{12} + \eta_{D1}\eta_{21}}{\eta_{D1}\eta_{D2} + \eta_{D2}\eta_{12} + \eta_{D1}\eta_{21}} \end{aligned} \tag{25}$$

follows, while $\gamma = \Delta N_2 / \Delta N_1$.

The above procedure is readily corroborated since the same result is produced if the directly derived functional relationship

$$E_2 = \frac{G_{P_1}\eta_{12}}{\eta_{D1}\eta_{D2} + \eta_{D2}\eta_{12} + \eta_{D1}\eta_{21}} \tag{26}$$

is used and the partial derivate $\partial E_2/\partial \eta_{12}$ is calculated.

3.1. Generic four subsystem model

In general the CLFs depend on frequency such that predictions of the sensitivity factor must be made for the frequency range of interest. For clarity the analyses herein are carried out just for one frequency. They can easily be extended, however, to frequency spectra.

Consider the four subsystem SEA model depicted in Fig. 1. The subsystems are assumed to have the same number of modes and damping loss factor, but the CLFs differ between the systems as indicated in the figure. To enable comparisons, the dissipation loss factors and CLFs are equal to those used by Craik [2]. The analysis is independent of the values of the parameters. The engineering task is to minimize the vibrational energy transmission from the source (subsystem 1) to the receiver (subsystem 4). By applying Craik's path-by-path analysis for the relevant paths one finds that

$$P_{1-2-4} = 0.16, \quad P_{1-2-3-4} = 1.43 \times 10^{-4},$$

$$P_{1-3-4} = 4.4 \times 10^{-5}, \quad P_{1-3-2-4} = 1.43 \times 10^{-4}. \tag{27}$$

The parameter P_{i-j} is the amount of energy transmitted along the path $i-j$. Upon calculating the sensitivity factors the following values are obtained:

$$\begin{aligned} S_{4,12} = 0.38, \quad & S_{4,13} = -4.5 \times 10^{-3} \\ & S_{4,23} = -3.7 \times 10^{-2}, \quad S_{4,24} = 0.38, \\ & \qquad\qquad\quad S_{4,34} = -4.5 \times 10^{-3}. \end{aligned} \tag{28}$$

A large value of $S_{j,\mu\nu}$ means that $\eta_{\mu\nu}$ has a large influence on the energy in subsystem j. Moreover, the energy in subsystem 4 is estimated to be $E_4 = 1.4 \times 10^{-3}$. To minimize the energy in subsystem 4 one possibility is to strengthen the coupling between subsystems 2 and 3. The negative value of $S_{4,23}$ indicates that an enhancement will reduce the energy transmitted to subsystem 4. Upon assuming that the CLF factor between subsystems 2 and 3 can be easily increased to a value of $\hat{\eta}_{23} = 10^{-3}$, this option should be investigated. By calculating the estimated effect from Eq. (9), the result is $\Delta E_4 = -5.1 \times 10^{-3}$, which of course is unphysical. Not surprisingly this estimation fails, since a change of the initial CLFs with a factor of 100 is not small. A recalculation of the whole system results in

$$P_{1-2-4} = 0.038, \quad P_{1-2-3-4} = 3.4 \times 10^{-4},$$

$$P_{1-3-4} = 4.5 \times 10^{-6} \quad P_{1-3-2-4} = 3.4 \times 10^{-4}. \tag{29}$$

This shows that the effect is that more energy is transmitted over the long lossy paths 1–2–3–4 and 1-3-2-4 and the energy in subsystem 4 is reduced to $E_4 = 1.0 \times 10^{-3}$. The new sensitivity factors are

$$\begin{aligned} S_{4,12} = 0.4, \quad & S_{4,13} = 3.5 \times 10^{-3}, \\ & S_{4,23} = -2.7 \times 10^{-2}, \quad S_{4,24} = 0.4, \\ & \qquad\qquad\quad S_{4,34} = 3.5 \times 10^{-3}. \end{aligned} \tag{30}$$

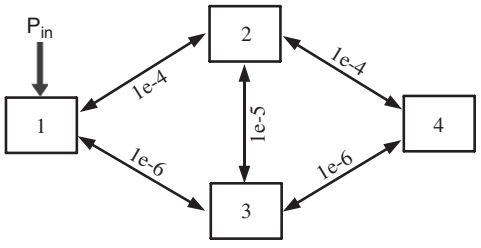

Fig. 1. A four subsystem SEA model with the coupling loss factors indicated.

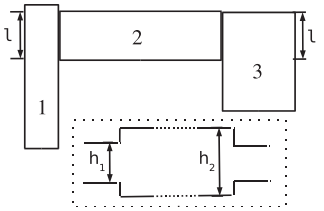

Fig. 2. Set-up of the SEA system.

showing that all sensitivities but $S_{4,23}, S_{4,12}$ and $S_{4,24}$ are reduced numerically and that $S_{4,13}$ and $S_{4,34}$ have changed sign.

In the next step the CLF between subsystem 1 and 2 is halved to $\hat{\eta}_{12} = 5 \times 10^{-5}$. The estimated effect from Eq. (9) is $\Delta E_4 = -2 \times 10^{-4}$. Recalculating the whole system results in $E_4 = 7.16 \times 10^{-4}$, which is just slightly less than the assessment $E_4 = \hat{E}_4 + \Delta E_4 = 8 \times 10^{-4}$. The new paths are

$$P_{1-2-4} = 0.02, \quad P_{1-2-3-4} = 1.8 \times 10^{-4},$$

$$P_{1-3-4} = 4.5 \times 10^{-6} \quad P_{1-3-2-4} = 3.6 \times 10^{-4}. \tag{31}$$

From the new sensitivity values, the interesting observation can be made that the best strategy is a further reduction of η_{12}.

$$\begin{aligned} S_{4,12} = 0.57, \quad & S_{4,13} = 1.0 \times 10^{-2}, \\ & S_{4,23} = -2.8 \times 10^{-2}, \quad S_{4,24} = 0.4, \\ & \qquad\qquad\qquad\quad S_{4,34} = 3.5 \times 10^{-3}. \end{aligned} \tag{32}$$

Without doubt, the path-by-path analysis is very useful for understanding the flow of energy in a system. The advantage of the sensitivity factors is that they provide a more detailed view. Moreover, the approach offers the possibility to weigh the sensitivity factors with some cost aspect (financial, mass, etc.) and achieving a rational basis for a redesign (Fig. 2).

4. Examples: propagation of uncertainties

To examine the variance due to, for example, an uncertain coupling length between two subsystems, the example in Section 3 is adapted.

Two plates that are edge coupled over the length l constitute the subsystems 1 and 2. The resulting variance of the energy in subsystem 2 is studied when the coupling length has an uncertainty Δl. The CLFs are [13]

$$\eta_{12} = \Delta N_2 / \Delta N_1 \eta_{21} = \tau_{12} \frac{c_{g1}}{\omega} \frac{l}{\pi S_1}. \tag{33}$$

With Eq. (33) $p, q = 1, 2$

$$\frac{\partial c_{pq}}{\partial l} = \begin{cases} -\frac{1}{l}\eta_{pq}, & p \neq q, \\ \frac{1}{l}\sum_{i \neq p} \eta_{pi}, & p = q. \end{cases} \tag{34}$$

In view of Eq. (22), this leads to the normalized standard deviation

$$\frac{\sigma_{E_j}}{E_j} = \left| \left(1 - \frac{\eta_{D2}\eta_{12} + \eta_{D1}\eta_{21}}{\eta_{D1}\eta_{D2} + \eta_{D2}\eta_{12} + \eta_{D1}\eta_{21}} \right) \frac{\Delta l}{l} \right|. \tag{35}$$

The result is analogous to that in Section 3 and expresses the implication of the uncertain variable x_u on the subsystem energy. It should be noted that x_u can be any variable provided that the relationship $\partial c_{ij}/\partial x_u$ is known.

4.1. Cascaded subsystems

With the derivation corroborated, a Matlab script was written to examine a generic configuration. A third subsystem is introduced with a second uncertain input quantity; the thickness ratio ϕ between plates 2 and 3. The system is sketched in figure

$$\tau_{23} = \left(\frac{\phi^{-5/4} + \phi^{-3/4} + \phi^{3/4} + \phi^{5/4}}{0.5\phi^{-2} + \phi^{-1/2} + 1 + \phi^{1/2} + 0.5\phi^{2}} \right)^{2}. \tag{36}$$

With the partial derivate $\partial\tau_{23}/\partial\phi$ and the implementation of the variance calculation given by Eq. (22), a numerical example is calculated. Fig. 3 shows the energy in the third subsystem with the standard deviation. The dashed lines show the impact of the uncertainty of the thickness ratio ϕ only and the grey area shows the total uncertainty due to the uncertainties in coupling length and cross-sectional thicknesses. Both normalized

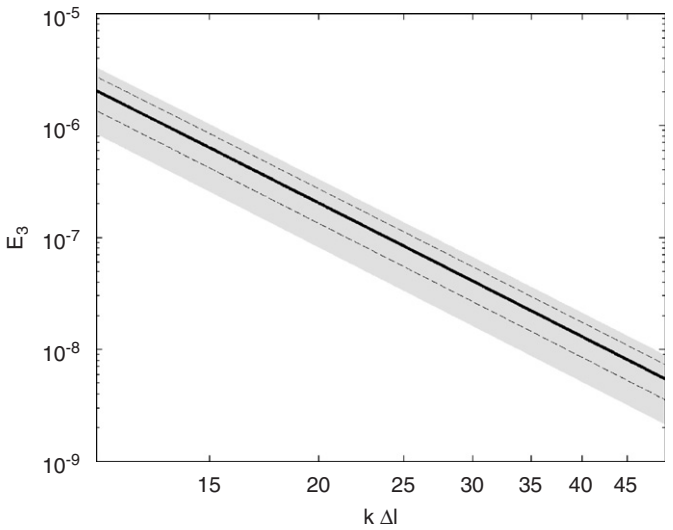

Fig. 3. Energy in subsystem 3 with standard deviation.

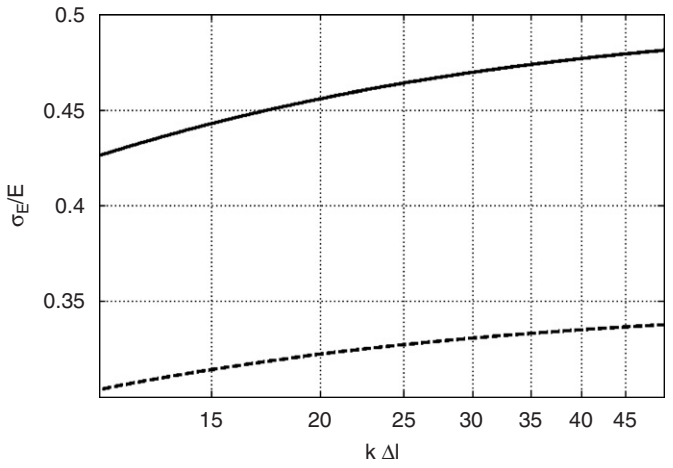

Fig. 4. Normalized standard deviation in subsystem energy due to uncertainties in coupling length (solid) and thickness ratio (dashed), both indicated normalized standard deviations 0.1.

uncertainties are 10%. The frequency dependent variance is plotted in Fig. 4. It shows that the impact of the thickness ratio uncertainty is roughly 2/3 of that of the coupling length. The variance is slightly increasing with frequency, an effect that is stated also by Culla in Ref. [8], while model uncertainties usually decrease with frequency. An investigation of uncertainties in the input parameters, however, can result in different tendencies. The analysis of the variance reveals the influence of system parameters.

5. Concluding remarks

An algorithm to calculate a factor expressing the sensitivity of a subsystems energy to CLFs is presented. Based on this algorithm a method is devised for the analysis of the significance of the various paths for the energy flow in a system. The flexibility of the method and the detailed insight gained into the influence of parameters of the system can be very helpful in design. Though demonstrated viable in a few examples, a practical application for real systems is required to verify the usefulness of the method.

Furthermore, it is shown that the uncertainty in an SEA prediction, resulting from uncertainties in input data or system parameters, can be assessed analytically. The prerequisite is that a functional relationship must be available between the CLF and the uncertain quantities. Although the uncertainties herein are assumed to have a Gaussian distribution this is no limitation. Provided the uncertainty variance can be properly assessed, the estimation of the uncertainty propagation will remain valid also for other distributions.

Appendix A

To calculate the derivative of the a inverse matrix, it is necessary to establish a variant definition of the sub-determinant. The sub-matrix $\mathbf{M}_{\mu,v}$ is usually constructed by deleting the (μ, v) row and column. The sub-determinant is calculated as the determinant of this sub-matrix with the leading sign given by $(-1)^{\mu+v}$. This is the formal way to calculate the inverse matrix $\mathbf{M}^{-1} = m_{\mu v}^{-1} = \mathbf{M}_{v\mu}/\det\mathbf{M}$. The subsub-matrix $M_{\mu v_{ij}}$ is developed by setting the (μ, v) element to unity and the rest of the elements in the μ-row and the j-row to zero. The leading sign follows that of the determinant. In such a way, is established the subsub-matrix $M_{\mu v_{ij}}$

$$\mathbf{M}_{\boldsymbol{\mu v_{ij}}} = \begin{pmatrix} m_{11} & \cdots & m_{\mu 1}=0 & \cdots & m_{Nj}=0 & \cdots & m_{N1} \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{1v} & & m_{\mu v}=1 & & 0 & & m_{Nv} \\ m_{1j} & & 0 & & m_{ij}=1 & & m_{Nj} \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{1N} & \cdots & m_{\mu N}=0 & \cdots & m_{iN}=0 & \cdots & m_{NN} \end{pmatrix}, \tag{A.1}$$

just by repeating the procedure. Only for $(\mu = i, v = j)$, also the (i,j) element is zero and so $\det M_{ij_{ij}} = 0$. To show that the sub-matrix definition is useful for the calculation of the derivative of the determinant, a step-by-step procedure follows for the development of the derivative of the determinant of an arbitrary matrix. Required for the procedure is a permutation symbol ε given by

$$\varepsilon^{k_1,\ldots,k_n} = \begin{cases} 1 & \text{for even permutations of } 1,\ldots,n, \\ -1 & \text{for uneven permutations of } 1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

Accordingly, the derivative of the determinant with respect to its elements is obtained as

$$\begin{aligned} \frac{\partial \det\mathbf{M}}{\partial m_{ij}} &= \frac{\partial}{\partial m_{ij}} \varepsilon^{k_1,\ldots,k_n} m_{1k_1},\ldots,m_{nk_n} = \varepsilon^{k_1,\ldots,k_n} \frac{\partial}{\partial m_{ij}} m_{1k_1},\ldots,m_{nk_n} \\ &= \varepsilon^{k_1,\ldots,k_n} m_{1k_1},\ldots,m_{i-1k_{i-1}},\delta_{jk_i},m_{i+1k_{i+1}},\ldots,m_{nk_n} \\ &= \det\mathbf{M}_{ij}. \end{aligned} \tag{A.3}$$

When calculating the derivative of the sub-determinant $\det \mathbf{M}_{\mu\nu}$ the definition above is used by inserting the Kronecker-δ at the ν-row

$$\begin{aligned}\frac{\partial \det \mathbf{M}_{\mu\nu}}{\partial m_{ij}} &= \varepsilon^{k_1,\ldots,k_n} \frac{\partial}{\partial m_{ij}} m_{1k_1}, \ldots, m_{\mu-1k_{\mu-1}}, \delta_{\nu k_\mu}, m_{\mu+1k_{\mu+1}}, \ldots, m_{nk_n} \\ &\overset{\mu\neq i}{=} \varepsilon^{k_1,\ldots,k_n} m_{1k_1}, \ldots, m_{i-1k_{i-1}}, \delta_{jk_i}, m_{i+1k_{i+1}}, \ldots, m_{\mu-1k_{\mu-1}}, \delta_{\nu k_\mu}, m_{\mu+1k_{\mu+1}}, \ldots, m_{nk_n} \\ &= \det \mathbf{M}_{\mu\nu_{ij}}. \end{aligned} \tag{A.4}$$

If $\mu = i$ then $\partial \det \mathbf{M}_{\mu\nu}/\partial m_{ij}$ equals zero and if $\nu = j$, it follows that $\partial \det \mathbf{M}_{\mu\nu}/\partial m_{ij}$ is zero because the derivative $\partial \delta_{jk_i}/\partial m_{ij} = 0$.

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